EVASION GAME PROBLEM WITH INFORMATION LAG

PMM Vol. 35, No. 4, 1971, pp. 575-583
V. S. SHISHMAKOV
(Sverdlovsk)
(Received March 30, 1971)

We consider the game problem of evading contact with a given set under the condition that the second player receives information on the state of the game after a constant delay. The paper is adjacent to [1–4]. On the basis of extremal construction [5] we form the optimal control of the second player. We consider sufficient conditions under which evasion is possible during a finite interval of time.

1. Let the motion of a conflict-controlled system be described by the differential equation

$$dx/dt = A(t)x + B(t)u - C(t)v, \quad t \geqslant t^\circ - \eta, \quad x[t^\circ - \eta] = x^\circ \tag{1.1}$$

Here η is a positive constant, x is an n-dimensional phase vector of the system, u and v are control vectors of dimension k and l, respectively, $A(t)$, $B(t)$, $C(t)$ are matrices of appropriate dimensions, continuous in t. The realization $u[t]$ and $v[t]$ of the controls at each instant $t \geqslant t^\circ - \eta$ are subject to the constraints

$$u[t] \in U_t, \qquad v[t] \in V_t \tag{1.2}$$

where U_t and V_t are bounded, closed and convex sets which vary continuously as t changes. Let there also be given a convex, closed and bounded set $M \subset E_n$. We consider the following game. The first player strives to bring motion (1.1) onto set M. The second player strives to avoid contact with M for as long as possible. Here, at each instant $t \geqslant t^\circ$ the second player can form his own control on the basis of the state of system (1.1) at the instant $t - \eta$. The control $v[\tau]$ is taken as given for $\tau \in [t^\circ - \eta, t^\circ)$. In this paper the situation is considered from the second player's viewpoint. Therefore, we shall not state explicitly the information available to the first player, assuming it to be as complete as desired.

Definition 1.1. Every summable function satisfying the condition $u(t) \in U_t$ ($v(t) \in V_t$) is called an admissible program control $u(t)$ ($v(t)$).

Definition 1.2. The collection $\{t, x[t-\eta], v(\cdot|t)\}$ us called the position of the game at the instant $t \geqslant t^\circ$. Here $v(\cdot|t)$ denotes a realization of the second player's control on the semi-interval $[t - \eta, t)$. The value of the control $v(\cdot|t)$ at the instant $\tau \in [t-\eta, t)$ is denoted by $v(\tau|t)$.

Definition 1.3. The rule which associates with each position $\{t, x, v(\cdot|t)\}$ a set $V(t, x, v(\cdot|t))$ satisfying the following Conditions:

1. the set $V(t, x, v(\cdot|t))$ belongs to V_t and is convex and closed ;
2. the set $V(t, x, v(\cdot|t))$ is upper semicontinuous relative to inclusion with respect to t, x and is upper ω-semicontinuous relative to inclusion with respect to $v(\cdot|t)$.

is called an admissible strategy V of the second player.

Condition 2 signifies the following. If the Conditions

$$v^k \to v^*, \quad t_k \to t_*, \quad x_k \to x_*, \quad v_k(\cdot \mid t_k) \underset{\omega}{\to} v_*(\cdot \mid t_*)$$

$$v^k \in V(t_k, x_k, v_k(\cdot \mid t_k))$$

are fulfilled, there the inclusion

$$v^* \in V(t_*, x_*, v_*(\cdot \mid t_*))$$

holds, where the symbol $\underset{\omega}{\to}$ denotes the weak convergence of the sequence of functions $v_k(\tau|t_k) = v_k(\xi)$ (here for convenience we have made the substitution $\xi = \tau + \eta - t_k$, $\tau \in [t_k - \eta, t_k)$), considered as elements of the space $L_2[0, \eta]$. As regards the first player, we shall take it that his strategy at each instant $t \geqslant t^\circ - \eta$ is characterized by the set U_t. Such a definition of the strategy allows us to cover any method by which the first player forms his control, generating a summable realization $u(t)$.

Definition 1.4. Every absolutely-continuous function $x[t]$ ($t \in [t^\circ - \eta, \vartheta]$) which for almost all t satisfies the following inclusion

$$dx[t]/dt \in A(t)x[t] + B(t)U_t - C(t)V_*(t, x[t-\eta], v(\cdot \mid t)) \qquad (1.3)$$

where

$$V_*(t, x, v(\cdot \mid t)) \doteq \begin{cases} v^\circ(t/t^\circ) & \text{for } t \in [t^\circ - \eta, t^\circ) \\ V(t, x, v(\cdot \mid t)) & \text{for } t \in [t^\circ, \vartheta] \end{cases}$$

is called a motion of system (1.1) during the interval $[t^\circ - \eta, \vartheta]$. The expression on the right-hand side of (1.3) should be understood as an algebraic sum of the sets. The existence of a solution of the differential equations in contingencies (1.3) can be shown by taking the corresponding Euler polynomial lines [6] to the limit.

Problem 1.1. We are required to construct an admissible strategy $V^\circ(t, x, v(\cdot|t))$ ($t \geqslant t^\circ$) for the second player, such that the condition $x[t] \notin M$ for all $t \in [t^\circ, \vartheta)$, where ϑ is some specified instant, is fulfilled for any motion of system (1.3) of differential equations in contingencies.

2. Let us consider an auxiliary program problem which will be used to construct the second player's optimal strategy $V^\circ(t, x, v(\cdot|t))$. In the phase coordinate space $\{x\}$, at a selected instant ϑ we associate with every instant $t \leqslant \vartheta$ and with an admissible control $v(\tau|t)$ ($\tau \in [t - \eta, t)$) a set $W(v(\cdot|t), t, \vartheta)$ defined as follows.

Definition 2.1. The set $W(v(\cdot \mid t), t, \vartheta)$ is the set of all points w possessing the following property: for any admissible control $v(\tau)$ ($\tau \in [t, \vartheta]$) we can choose an admissible control $u(\tau)$ ($\tau \in [t - \eta, \vartheta]$) such that the pair of controls $v^*(\tau)$ and $u(\tau)$, where

$$v^*(\tau) = \begin{cases} v(\tau \mid t) & \text{for } \tau \in [t - \eta, t) \\ v(\tau) & \text{for } \tau \in [t, \vartheta] \end{cases}$$

takes system (1.1) from the point $x[t - \eta] = w$ into some state $x[\vartheta] \in M$.

In correspondence with Definition 2.1 we can specify the set $W(v(\cdot \mid t), t, \vartheta)$ as the collection of vectors w for which the inclusion

$$G^{(2)}(t, \vartheta) \subset G^{(1)}(t - \eta, \vartheta) + X(\vartheta, t)\,[X(t, t - \eta)\,w - g] - M \qquad (2.1)$$

is valid. Here $G^{(1)}(t, \vartheta)$ and $G^{(2)}(t, \vartheta)$ are the first and second player's reachable regions from the state $x[t] = 0$ at the instant $\vartheta \geqslant t$, defined, respectively, as the sets of all points $x^{(1)}$ and $x^{(2)}$ satisfying the relations

$$x^{(1)} = \int_t^{\vartheta} X(\vartheta, \tau) B(\tau) u(\tau)\, d\tau, \qquad u(\tau) \in U_\tau$$

$$x^{(2)} = \int_t^{\vartheta} X(\vartheta, \tau) C(\tau) v(\tau)\, d\tau, \qquad v(\tau) \in V_\tau$$

where $X(\vartheta, t)$ is the fundamental matrix of system (1.1), while $g = g(t)$ is a vector defined by the expression

$$g(t) = \int_{t-\eta}^{t} X(\vartheta, \tau) C(\tau) v(\tau \mid t)\, d\tau \tag{2.2}$$

We list certain properties of the sets $W(v(\cdot \mid t), t, \vartheta)$, useful subsequently, which ensue directly from Definition 2.1 and relation (2.1).

Assertion 2.1. The sets $W(v(\cdot \mid t), t, \vartheta)$ are convex, closed and bounded.

Assertion 2.2. If the position $\{\vartheta, x, v(\cdot \mid \vartheta)\}$ is such that

$$x \notin W(v(\cdot \mid \vartheta), \vartheta, \vartheta)$$

then any motion starting from the point $x = x[\vartheta - \eta]$ does not hit onto M at the instant $t = \vartheta$.

From relation (2.1) it follows that the point w belongs to the set $W(v(\cdot \mid t), t, \vartheta)$ if and only if the condition

$$\max [\rho_2(s, t, \vartheta) - \rho_1(s, t-\eta, \vartheta) - \rho_{-M}(s) + s' X(\vartheta, t) g - s' X(\vartheta, t - \eta) w] \leqslant 0, \quad \|s\| = 1 \tag{2.3}$$

is fulfilled. Here s is a unit vector, $\rho_1(s, t, \vartheta)$, $\rho_2(s, t, \vartheta)$, $\rho_{-M}(s)$ are the support functions [5] of the sets $G^{(1)}(t, \vartheta)$, $G^{(2)}(t, \vartheta)$, $-M$.

Suppose that we are given an interval $[t^\circ, \vartheta^*]$. Let us take a position $\{t, x, v(\cdot \mid t)\}$ such that $x \notin W(v(\cdot \mid t), t, \vartheta)$ and with it and with the instant $\vartheta \in [t, \vartheta^*]$ let us associate a quantity $\varepsilon(t, x, g, \vartheta)$ defined by the expression

$$\varepsilon(t, x, g, \vartheta) = \max_s [\rho_2(s, t, \vartheta) - \rho_1(s, t-\eta, \vartheta) - \rho_{-M}(s) + s' X(\vartheta, t) g - s' X(\vartheta, t-\eta) x], \quad \|s\| = 1 \tag{2.4}$$

It is evident that the quantity $\varepsilon(t, x, g, \vartheta)$ is positive when $x \notin W(v(\cdot \mid t), t, \vartheta)$. We extend the definition of the function $\varepsilon(t, x, g, \vartheta)$ by setting it equal to zero when $x \in W(v(\cdot \mid t), t, \vartheta)$. It is not difficult to show that the following property holds.

Assertion 2.3. In the region $\varepsilon(t, x, g, \vartheta) > 0$ the function $\varepsilon(t, x, g, \vartheta)$ satisfies a Lipschitz condition in the argument ϑ, i.e. the inequality

$$\varepsilon(t, x, g, \vartheta'') - \varepsilon(t, x, g, \vartheta') < \lambda |\vartheta'' - \vartheta'| \tag{2.5}$$

is valid, where ϑ', ϑ'' are arbitrary instants from the interval $[t, \vartheta^*]$ and λ is some positive constant.

In what follows we shall take as fulfilled the next condition.

Condition 2.1. If the position $\{t, x, v(\cdot \mid t)\}$ and the instant $\vartheta \in [t, \vartheta^*]$ are such that $x \notin W(v(\cdot \mid t), t, \vartheta)$, then the maximum in the right-hand side of expression (2.4) is reached on the unit vector $s = s^\circ(t, x, g, \vartheta)$.

We can show (just as was done in [2]) that when Condition 2.1 is fulfilled the following assertions are valid.

Assertion 2.4. In the region $\varepsilon(t, x, g, \vartheta) > 0$ the functions $s^\circ(t, x, g, \vartheta)$ and $\varepsilon(t, x, g, \vartheta)$ are continuous functions of their arguments.

Assertion 2.5. In the region $\varepsilon(t, x, g, \vartheta) > 0$ there hold the following relations:

$$\operatorname{grad}_x \varepsilon(t, x, g, \vartheta) = -X'(\vartheta, t-\eta)\cdot s^\circ(t, x, g, \vartheta) \tag{2.6}$$

$$\operatorname{grad}_g \varepsilon(t, x, g, \vartheta) = X'(\vartheta, t)\cdot s^\circ(t, x, g, \vartheta) \tag{2.7}$$

$$\partial\varepsilon(t, x, g, \vartheta)/\partial t = \mu_1(s^\circ(t, x, g, \vartheta), t-\eta, \vartheta) - \mu_2(s^\circ(t, x, g, \vartheta), t, \vartheta) + s^{\circ\prime}(t, x, g, \vartheta)\cdot X(\vartheta, t-\eta)\cdot A(t-\eta)x - s^{\circ\prime}(t, x, g, \vartheta)X(\vartheta, t)\cdot A(t)g \tag{2.8}$$

where the functions $\mu_i(s, \tau, \vartheta)$ $(i = 1, 2)$ are defined by the relations

$$\mu_1(s, \tau, \vartheta) = \max_u s' X(\vartheta, \tau) B(\tau) u, \quad u \in U_\tau \tag{2.9}$$

$$\mu_2(s, \tau, \vartheta) = \max_v s' X(\vartheta, \tau) C(\tau) v, \quad v \in V_\tau$$

3. Suppose that the second player counts on avoiding the contact of point $x[t]$ with set M during the interval $[t^\circ, \vartheta^*]$. Consider an open domain D of the variables t, x, g, defined by the inequality

$$\min_\vartheta \varepsilon(t, x, g, \vartheta) > 0 \quad (\vartheta \in [t, \vartheta^*])$$

In domain D we construct the Liapunov function

$$L(t, x, g) = \int_t^{\vartheta^*} \varepsilon^{-1}(t, x, g, \vartheta)\, d\vartheta \tag{3.1}$$

Now, for each position $\{t, x, v(\cdot\,|\,t)\}$ we define a certain strategy $V^*(t, x, v(\cdot\,|\,t))$ of the second player in the following way:

$$V^*(t, x, v(\cdot\,|\,t)) = \begin{cases} V^{(e)} & \text{for } \min_\vartheta \varepsilon(t, x, g, \vartheta) > 0,\ \vartheta \in [t, \vartheta^*] \\ V_t & \text{for } \min_\vartheta \varepsilon(t, x, g, \vartheta) = 0,\ \vartheta \in [t, \vartheta^*] \end{cases} \tag{3.2}$$

where $V^{(e)}$ is the set of vectors v_e satisfying the relation

$$(\operatorname{grad}_g L(t, x, g))' C(t) v_e = \min_v (\operatorname{grad}_g L(t, x, g))' C(t) v, \quad v \in V_t \tag{3.3}$$

By using relations (3.1), (2.6)–(2.8), the continuity of function $s^\circ(t, x, g, \vartheta)$, and the continuity of the matrix $X(t, \tau)$, we can show that the function $L(t, x, g)$ is continuously differentiable in domain D.. Hence, from the continuity of matrix $C(t)$ and of the set V_t, with due regard to expression (2.2) for $g = g(t)$, it follows [6] that the strategy $V^*(t, x, v(\cdot\,|\,t))$ is an admissible strategy for the second player. We write out the expression for the total time derivative of function $L(t, x, g)$ relative to system (1.1) and relation (2.2) in the following form:

$$dL/dt = \Phi(t, x, g, u, v) = (\operatorname{grad}_x L(t, x, g))' B(t-\eta) u + (\operatorname{grad}_g L(t, x, g))' C(t) v + S(t, x, g) \tag{3.4}$$

where

$$u \in U_{t-\eta}, \qquad v \in V_t$$

$$S(t, x, g) = (\operatorname{grad}_x L(t, x, g))' [A(t-\eta)x - C(t-\eta)v(t-\eta\,|\,t)] + (\operatorname{grad}_g L(t, x, g))' [A(t)g - X(t, t-\eta)C(t-\eta)v(t-\eta\,|\,t)] + \partial L(t, x, g)/\partial t$$

Let us consider a function $\Phi^\circ(t, x, g)$, defined in domain D,

$$\Phi^\circ(t,x,g) = \min_v \max_u \Phi(t,x,g,u,v) = \max_u \min_v \Phi(t,x,g,u,v) = \\ = \max_u (\operatorname{grad}_x L(t,x,g))' B(t-\eta) u + \min_v (\operatorname{grad}_g L(t,x,g))' C(t) v + \\ + S(t,x,g), \qquad u \in U_{t-\eta}, \qquad v \in V_t \tag{3.5}$$

We show that the following assertion is valid.

Lemma 3.1. Suppose that the initial position $\{t^\circ, x^\circ, v(\cdot|t^\circ)\}$ is such that $x^\circ \notin W(v(\cdot|t^\circ), t^\circ, \vartheta)$ for all $\vartheta \in [t^\circ, \vartheta^*]$. Suppose also that system (1.1) and the constraints (1.2) on the controls are such that the inequality

$$\Phi^\circ(t, x, g) \leqslant \varkappa L(t, x, g) \quad (\varkappa\text{-const}) \tag{3.6}$$

is fulfilled in domain D. Then, the strategy $V^*(t, x, v(\cdot|t))$ ensures that any motion of system (1.1), generated by the strategies $V^*(t, x, v(\cdot|t))$ and U_t, will evade contact with set M during the interval $[t^\circ, \vartheta^*]$.

Proof. Let $x_e[t]$ be an arbitrary motion of system (1.1), generated by the strategies U_t and $V^*(t, x, v(\cdot|t))$. By $\{t, x_e[t-\eta], v_e(\cdot|t)\}$ we denote the position consisting of the motion $x_e[t-\eta]$ and of the second player's control $v_e(\tau|t)$ $(\tau \in [t-\eta, t])$ that have been realized. We show that when $t \in [t^\circ, \vartheta^*]$ the point $x_e = x_e[t-\eta]$ does not fall into any one of the sets $W(v_e(\cdot|t), t, \vartheta)$ $(\vartheta \in [t, \vartheta^*])$. Hence, by virtue of Assertion 2.2 it follows that $x_e[t] \notin M$ when $t \in [t^\circ, \vartheta^*]$.

We assume the contrary. Suppose that at some instant $t^* \in [t^\circ, \vartheta^*]$ a position $\{t^*, x_e[t^*-\eta], v_e(\cdot|t^*)\}$ is realized such that for the first time $x_e[t^*-\eta] \in W(v_e(\cdot|t^*), t^*, \vartheta')$, where ϑ' is some instant from the interval $[t^*, \vartheta^*]$. This signifies that

$$\varepsilon(t^*, x_e[t^*-\eta], g_e(t^*), \vartheta') = 0$$

Let us take a sequence of positions $\{t^n, x_e[t^n-\eta], v_e(\cdot|t^n)\}$, where $t^n < t^*, t^n \to t^*$ as $n \to \infty$. Then the sequence of numbers

$$\varepsilon^n = \varepsilon(t^n, x_e[t^n-\eta], g_e(t^n), \vartheta') > 0$$

tends to zero by virtue of the continuity of the function $\varepsilon(t, x, g, \vartheta)$. By Assertion 2.3, for any $\vartheta \in [t^n, \vartheta^*]$ we can write down the inequality

$$\varepsilon(t^n, x_e[t^n-\eta], g_e(t^n), \vartheta) < \varepsilon^n + \lambda|\vartheta - \vartheta'|$$

Hence we have the estimate

$$L(t^n, x_e[t^n-\eta], g_e(t^n)) \geqslant \int_{t^n}^{\vartheta'} [\varepsilon^n + \lambda|\vartheta-\vartheta'|]^{-1} d\vartheta,$$

from which it follows that

$$L(t^n, x_e[t^n-\eta], g_e(t^n)) \to \infty$$

as

$$n \to \infty$$

On the semi-interval $[t^\circ, t^*)$ let us treat the function $L(t, x_e[t-\eta], g_e(t))$ along the change in position $\{t, x_e[t-\eta], v_e(\cdot|t)\}$ as a time function $L(t)$. Since the function $L(t, x, g)$ is continuously differentiable and the functions $x_e[t]$ and $g_e[t]$ are absolutely continuous, the derivative

$$dL(t)/dt = \Phi(t) \tag{3.7}$$

exists almost everywhere on $[t^\circ, t^*)$, where $\Phi(t)$ is a realization of the function $\Phi(t, x_e[t-\eta], g_e(t), u[t-\eta], v_e[t])$ as a function of time. From relations (3.3)-(3.7) we obtain the estimate

$$dL(t)/dt \leqslant \varkappa L(t) \qquad \text{for } t \in [t^\circ, t^*),$$

from which follows the boundedness of the function $L(t)$ on the semi-interval $[t^\circ, t^*)$. Thus, we obtain a contradiction; on the one hand, the function $L(t)$ is bounded on $[t^\circ, t^*)$, on the other hand, the sequence $\{L(t^n)\}$ increases unboundedly as $t^n \to t^*$ $(t^n < t^*)$. This contradiction proves Lemma 3.1.

4. Let us determine the constraints on the players' controls and the second player's strategy which ensure that the motion $x[t]$ evades contact with set M during some semi-interval $[t^\circ, \vartheta^\circ)$. We introduce the definition of the absorption instant.

Definition 4.1. The smallest value of the parameter ϑ for which the inclusion $x \in W(v(\cdot|t), t, \vartheta)$ holds is called the absorption instant $\vartheta^\circ(t, x, v(\cdot|t))$ corresponding to the game position $\{t, x, v(\cdot|t)\}$.

Let an initial position $\{t^\circ, x^\circ, v(\ |t^\circ)\}$ be given and let ϑ° be the absorption instant corresponding to it. We construct the second player's strategy $V_\alpha^\circ(t, x, v(\cdot|t))$

$$V_\alpha^\circ(t, x, v(\cdot\,|\,t)) = \begin{cases} V^{(e)} & \text{for } \min_\vartheta \varepsilon(t, x, g, \vartheta) > 0 \\ V_t & \text{for } \min_\vartheta \varepsilon(t, x, g, \vartheta) = 0 \end{cases} \tag{4.1}$$

where $\vartheta \in [t, \vartheta^\circ - \alpha]$, α is a positive constant. In Lemma 3.1 we set $\vartheta^* = \vartheta^\circ - \alpha$. Then from Definitions 2.1 and 4.1 it follows that the first hypothesis of Lemma 3.1 is satisfied for any $\alpha \in (0, \vartheta^\circ - t^\circ]$, i.e. the initial position satisfies the condition $x^\circ \notin W(v(\cdot|t^\circ), t^\circ, \vartheta)$ for all $\vartheta \in [t^\circ, \vartheta^\circ - \alpha]$.

Let us now assume that system (1.1) and the constraints (1.2) on the players' controls are such that the condition

$$X(t, t-\eta)\, B(t-\eta)\, U_{t-\eta} = C(t)\, V_t + D_t \qquad (t \in [t^\circ, \vartheta^\circ]) \tag{4.2}$$

is fulfilled, where D_t is some convex set. We show that Condition 1.2 is satisfied when relation (4.2) is fulfilled. Consider the function

$$\rho(s, t-\eta, \vartheta) = \rho_1(s, t-\eta, \vartheta) - \rho_2(s, t, \vartheta) \tag{4.3}$$

This equality can be written in the form [5]

$$\rho(s, t-\eta, \vartheta) = \int_{t-\eta}^{\vartheta} \mu_1(s, \tau, \vartheta)\, d\tau - \int_t^{\vartheta} \mu_2(s, \tau, \vartheta)\, d\tau =$$

$$= \int_t^{\vartheta} \mu_1(s, \tau-\eta, \vartheta)\, d\tau - \int_t^{\vartheta} \mu_2(s, \tau, \vartheta)\, d\tau + \int_{\vartheta-\eta}^{\vartheta} \mu_1(s, \tau, \vartheta)\, d\tau$$

where $\mu_i(s, \tau, \vartheta)$ $(i = 1,2)$ are defined by expressions (2.9). Hence, and from (4.2) it follows that

$$\rho(s, t-\eta, \vartheta) = \int_t^{\vartheta} \mu(s, \tau, \vartheta)\, d\tau + \int_{\vartheta-\eta}^{\vartheta} \mu_1(s, \tau, \vartheta)\, d\tau \tag{4.4}$$

where

$$\mu(s, \tau, \vartheta) = \max_y s' X(\vartheta, \tau)\, y \quad \text{for } y \in D_\tau,\ \tau \in [t, \vartheta]$$

From expression (4.4) it follows that the difference (4.3) is function which is convex in s. Using this property we can show [5] that the maximum in expression (2.4) is reached on the unique vector $s = s^\circ(t, x, g, \vartheta)$, i.e. Condition 2.1 is fulfilled.

We now verify the satisfaction of the second hypothesis of Lemma 3.1. To do this we take into account the relations (2.6)–(2.8), (3.1), (3.4), (4.4) and we write the expression for the function

$$\Phi(t,x,g,u,v)=$$
$$=\int_t^{\vartheta^\circ-\alpha}\left\{\frac{s^\circ(t,x,g,\vartheta)X(\vartheta,t)[X(t,t-\eta)B(t-\eta)u-C(t)v]}{\varepsilon^2(t,x,g,\vartheta)}-\right.$$
$$\left.-\frac{\mu(s(t,x,g,\vartheta),t,\vartheta)}{\varepsilon^2(t,x,g,\vartheta)}\right\}d\vartheta-\varepsilon^{-1}(t,x,g,t) \qquad (4.5)$$

where

$$u\in U_{t-\eta},\qquad v\in V_t,\qquad t\in[t^\circ,\vartheta^\circ-\alpha]$$

Let u°, v° be a pair of vectors for which the equality

$$\Phi(t,x,g,u^\circ,v^\circ)=\Phi^\circ(t,x,g)$$

holds. By virtue of (4.2) there exists a vector v^* such that

$$X(t,\ t-\eta)B(t-\eta)u^\circ-C(t)v^*\in D_t$$

By virtue of (3.5) the inequality

$$\Phi^\circ(t,x,g)\leqslant\Phi(t,x,g,u^\circ,v^*)$$

is valid for the vector v^*. Hence and from (4.5), taking into account that the function $\varepsilon(t,x,g,t)$ is positive, we get that

$$\Phi^\circ(t,x,g)\leqslant 0$$

Thus the second hypothesis of Lemma 3.1 is satisfied. Using formula (4.5) and (3.3) we write the set $V^{(e)}$ defining the second player's strategy $V_\alpha^\circ(t,x,v(\cdot\,|\,t))$ as the collection of vectors v_e satisfying the condition

$$l_\alpha'(t,x,g)C(t)v_e=\max_v l_\alpha'(t,x,g)C(t)v,\quad v\in V_t \qquad (4.6)$$

where the vector $l_\alpha(t,x,g)$ is defined by the equality

$$l_\alpha(t,x,g)=\int_t^{\vartheta^\circ-\alpha}\frac{X'(\vartheta,t)s^\circ(t,x,g,\vartheta)}{\varepsilon^2(t,x,g,\vartheta)}\,d\vartheta \qquad (4.7)$$

Thus, the following assertion is valid.

Theorem 4.1. Suppose that the system of equations (1.1) and the constraints (1.2) on the controls are such that condition (4.2) is satisfied. Then the second player's strategy $V_\alpha^\circ(t,x,v(\cdot\,|\,t))$, given by (4.1), (4.6), (4.7), guarantees the evasion of motion $x[t]$ from contact with set M during the interval $[t^\circ,\vartheta^\circ-\alpha]$ for any $\alpha\in(0,\vartheta^\circ-t^\circ]$. On the other hand, it turns out that the second player cannot guarantee the evasion of system (1.1) from contact with set M during an interval larger than the semi-interval $[t^\circ,\vartheta^\circ)$.

To prove this we consider the first player's strategy defined at each instant $t-\eta$ $(t\in[t^\circ,\vartheta^\circ])$ by the set

$$U^\circ(t-\eta,x,v(\cdot\,|\,t))=\begin{cases}U^{(e)} & \text{for } x\notin W(v(\cdot\,|\,t),t,\vartheta^\circ)\\ U_t & \text{for } x\in W(v(\cdot\,|\,t),t,\vartheta^\circ)\end{cases}$$

where $U^{(e)}$ is the set of vectors u_e satisfying the relation

$$s^{\circ\prime}(t,\ x,\ g,\ \vartheta^\circ)\ X(\vartheta^\circ,\ t-\eta)\ u_\varepsilon =$$
$$= \max_u s^{\circ\prime}(t,\ x,\ g,\ \vartheta^\circ)\ X(\vartheta^\circ,\ t-\eta)\ B(t-\eta)\ u, \qquad u \in U_{t-\eta}$$

The strategy $U^\circ(t-\eta,\ x,\ v(\cdot\,|\,t))$ assumes that the first player knows at the instant $t-\eta$ the realization $v(\cdot\,|\,t)$ of the second player's control during the interval $[t-\eta,\ t)$. Such an availability to the first player of information on the future state of the game is, it is clear, not realistic. However, by virtue of the way Problem 1.1 was posed the second player is not ensured that the first player will not apply a realization of control $u(t)$ which during the game satisfies the inclusion $u(t-\eta) \in U^\circ(t-\eta,\ x,\ v(\cdot\,|\,t))$.

With the aid of arguments analogous to those in [1, 6] we can show that when Condition 2.1 is fulfilled the first player's strategy $U^\circ(t-\eta,\ x,\ v(\cdot\,|\,t))$ guarantees that system (1.1) is led onto set M no later than the instant ϑ°. Hence it follows that from the second player's point of view the instant ϑ° is the best instant for which Problem 1.1 can be solved.

Let us consider the case when the set M is a linear subspace of the space E_n. By $X^*(\vartheta,\ t)$ we denote the matrix whose columns are the projections of the column-vectors of matrix $X(\vartheta,\ t)$ onto the orthogonal complement of subspace M. Suppose that the condition

$$X^*(\vartheta, t)\, X(t, t-\eta)\, B(t-\eta)\, U_{t-\eta} = X^*(\vartheta, t)\, C(t)\, V_t + D(\vartheta, t) \tag{4.8}$$

is fulfilled for all $\vartheta \in [t^\circ,\ \vartheta^\circ]$, $t \in [t^\circ,\ \vartheta^\circ]$, where $D(\vartheta,\ t)$ is some convex set. Also suppose that for any $t \in [t^\circ,\ \vartheta^\circ]$, $u \in U_{t-\eta}$ we can choose a vector $v \in V_t$ such that the inclusion

$$X^*(\vartheta,\ t)\ [X(t,\ t-\eta)\ B(t-\eta)\ u - C(t)v] \in D(\vartheta,\ t) \tag{4.9}$$

holds for all $\vartheta \in [t,\ \vartheta]$. By $l_\alpha^*(t,\ x,\ g)$ we denote a vector defined by the formula

$$l_\alpha^*(t, x, g) = \int_t^{\vartheta^\circ-\alpha} \frac{X^*(\vartheta, t)\, s^\circ(t, x, g, \vartheta)}{\varepsilon^2(t, x, g, \vartheta)}\, d\vartheta \tag{4.10}$$

Just as in the case of a bounded set M, here we can show the validity of an assertion analogous to Theorem 4.1. Here, instead of the fulfillment of condition (4.2) we require the fulfillment of the weaker conditions (4.8) and (4.9), while the second player's strategy is determined by formulas (4.1), (4.6), (4.10).

The author thanks N. N. Krasovskii and A. I. Subbotin for their attention to the work.

BIBLIOGRAPHY

1. Krasovskii, N. N., Game problems of dynamics, II. Izv. Akad. Nauk SSSR, Tekh. Kibernetika №1, 1970.
2. Krasovskii, N. N. and Subbotin, A. I., Optimal strategies in a linear differential game. PMM Vol. 33, №4, 1969.
3. Pontriagin, L. S. and Mishchenko, E. F., The problem of one controlled object running away from another. Dokl. Akad. Nauk SSSR Vol. 189, №4, 1969.
4. Chernous'ko, F. L., A differential game with lagging information. Dokl. Akad. Nauk SSSR Vol. 188, №4, 1969.
5. Krasovskii, N. N., Game Problems on the Encounter of Motions. Moscow, "Nauka", 1970.
6. Shishmakov, V. S., A differential game of guidance under lagging information. Upravliaemye Sistemy №7, 1970.

Translated by N. H. C.